Some remarks on Liouville type theorems

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Some remarks on Liouville type theorems. **(English summary)**
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The authors present here elementary proofs of statements related to the Liouville theorem for the equation

$$-\nabla\cdot(A(x)\nabla u(x))+a(x)u=0$$

in $\mathcal{D}'(\mathbb{R}^k)$, where $a\in L^\infty_{\rm loc}(\mathbb{R}^k)$, $a(x)\geq 0$ and $A(x)=(a_{ij}(x))$ is a $k\times k$ uniformly elliptic matrix of bounded measurable coefficients. When $A(x)\equiv\delta_{ij}$ the equation is the so-called stationary Schrödinger equation [see Y. Pinchover, in *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, 329–355, Proc. Sympos. Pure Math., 76, Part 1, Amer. Math. Soc., Providence, RI, 2007; MR2310209 (2008e:35002); B. Simon, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447–526; MR0670130 (86b:81001a)]. When $a\equiv 0$ it is well known that every bounded solution has to be constant (see [L. C. Evans, *Partial differential equations*, Amer. Math. Soc., Providence, RI, 1998; MR1625845 (99e:35001); M. Meier, Manuscripta Math. **29** (1979), no. 2-4, 207–228; MR0545042 (80m:35024); J. Moser, Comm. Pure Appl. Math. **14** (1961), 577–591; MR0159138 (28 #2356)] and also [H. Berestycki, I. Capuzzo Dolcetta and L. Nirenberg, Topol. Methods Nonlinear Anal. **4** (1994), no. 1, 59–78; MR1321809 (96d:35041); M. Rigoli and A. G. Setti, NoDEA Nonlinear Differential Equations Appl. **9** (2002), no. 1, 15–36; MR1891293 (2002k:35096)] for some nonlinear versions). The case where $a\neq 0$ and $k\geq 3$ is very different and in this case nontrivial bounded solutions might exist. Many of the results in this paper are known in one form or another [see S. Agmon, in *Differential equations (Birmingham, Ala., 1983)*, 7–17, North-Holland, Amsterdam, 1984; MR0799327 (87a:35060); C. J. K. Batty, Math. Ann. **292** (1992), no. 3, 457–492; MR1152946 (93g:47050); W. Arendt, C. J. K. Batty and P. Bénilan, Math. Z. **209** (1992), no. 4, 511–518; MR1156433 (93i:47057); A. A. Grigor′yan, Trudy Sem. Petrovsk. No. 14 (1989), 66–77, 265–266; MR1001354 (90m:35050); Bull. Amer. Math. Soc. (N.S.) **36** (1999), no. 2, 135–249; MR1659871 (99k:58195); A. A. Grigor′yan and W. Hansen, Math. Ann. **312** (1998), no. 4, 659–716; MR1660247 (2000a:58092); Y. Pinchover, Differential Integral Equations **5** (1992), no. 3, 481–493; MR1157482 (93b:35035); R. G. Pinsky, *Positive harmonic functions and diffusion*, Cambridge Univ. Press, Cambridge, 1995; MR1326606 (96m:60179); Trans. Amer. Math. Soc. **360** (2008), no. 12, 6545–6554; MR2434298 (2009i:35058)], but the proofs presented here are based on simple self-contained PDE techniques. For instance, Liouville-type results are proved in the case where the growth of u is controlled (i.e., for r large $r^{-2}\int_{r\Omega\smallsetminus\frac{r}{2}\Omega}u^2dx\leq C$, with Ω being a bounded domain containing the origin), in the case of decay of $a(x)$ (i.e., for $|x|$ large $a(x)\geq\frac{c}{|x|^\beta}$, $\beta<2$ or $\beta=2$ and in such a case $a_{ij}\in C^1(\mathbb{R}^k\smallsetminus B(0,R))$ with $\partial_i(a_{ij}(x))\cdot x_j\leq D$ for $|x|>R$ and some $R>0$) and in the case where at infinity a has enough mass locally (i.e., $a(x)\geq a_0>0$ at infinity or $a(x)\geq a_p$ where a_p is a periodic function). When $A_{ij}\equiv\delta_{ij}$ bounded nontrivial solutions exist if $a\neq 0$ and $\int_{|x|>1}a(x)|x|^{-k+2}dx<\infty$ for $k\geq 3$, and do not exist if $a(x)\geq\overline{a}(|x|)$ for $|x|$ large with $\int^\infty r\overline{a}(r)dr=+\infty$.

{For the entire collection see MR2416199 (2009f:35002)}

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